Q/ If a function f defined on $[0,1]$ by:

$$f(x) = \begin{cases} x & , \text{ if } x \text{ is rational} \\ 1-x & , \text{ if } x \text{ is irrational} \end{cases}$$

then show that f takes every value between 0 and 1 (both inclusive) but continuous only at $x = 1/2$.

Proof:- Case (1) Let $x = a$ be a rational number

then LHL at $x = a$

$$= f(a-) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} [1-(a-h)]$$

$$= 1-a \quad \text{———} (1)$$

$\because a-h$ may be irrational,

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Similarly RHL at $x = a$

$= f(a+)$

$= \lim_{h\to 0} f(a+h) = \lim_{h\to 0} [1-(a+h)] = 1-a$ ---- (ii)

V.o.f^n: $= f(a) = a$ ------------ (iii)

For continuity at $x=0$: LHL = RHL = v of f^n

$\Rightarrow 1-a = 1-a = a \Rightarrow \boxed{a = \frac{1}{2}}$

$\Rightarrow f^n$ will be continous only at $x = \frac{1}{2}$.

Again:

$f(0) = 0$ —— (iv)

$f(1) = 1$ —— (v).

$\Rightarrow f$ takes value from 0 to 1.

Again. let $x = a$ is irrational number

then LHL at $x = a$

$= f(a-) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} (a-h) = a$ —— (vi)

RHL $= f(a+) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} (a+h) = a$ —— (vii)

V.o.f^n. $= f(a) = 1-a$ ———— (viii)

for continuity $a = a = 1-a \Rightarrow \boxed{a = \frac{1}{2}}$

which is not irrational number

$\Rightarrow f$ will not be continous at any irrational number.

H.P

Th. Show that, if a function $f(x)$ is continous in $[a,b]$, then for $\epsilon > 0$ the interval $[a,b]$ can be subdivided into a finite number of subintervals such that in each of which $|f(x_1) - f(x_2)| < \epsilon$ for any two points x_1 and x_2 in the same subinterval.

Proof:- Let us take contrary, that the theorem is not true in $[a,b]$.

$\Rightarrow$ After the subdivision of $[a,b]$, the theorem should not be true in at least one subdivided interval.

Now let $I_0 = [a,b]$ and subdivide into two equal parts $[a,c]$ & $[c,b]$, where $c = a + \left(\frac{b-a}{2}\right)$ i.e. $\frac{a+b}{2}$ i.e. middle part of $[a,b]$.

$\Rightarrow$ Theorem is false in either $[a,c]$ or $[c,b]$

let it be $[c,b]$, then rename it as $[a_1, b_1]$ and let $I_1 = [a_1, b_1]$ i.e. $a_1 = c$; $b_1 = b$.

$\Rightarrow$ Theorem is false in $[a_1, b_1]$

Again subdivide $[a_1, b_1]$ into two equal parts $[a_1, d]$, $[d, b_1]$ and theorem will again false in any one of them.

and after renaming it, let it be
$I_2 = [a_2, b_2]$.
Now continue this process of subdivision of $[a, b]$, we have a sequence of subintervals $I_1, I_2, I_3, I_4, \ldots, I_n \ldots$
where $I_n = [a_n, b_n]$ in which theorem

is not true.

Length $= |I_n| = \left|\frac{1}{2^n}(b-a)\right|$

The sequence of the subintervals $\langle I_n \rangle$ has the following properties:

(i) $I_0 \supset I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset$

(ii) length of $I_n = \left|\frac{1}{2^n}(b-a)\right|$

and $\lim\limits_{n \to \infty} I_n = 0$

(iii) Theorem is false in every I_n.

$\Rightarrow$ By the theorem of nested interval theorem $\bigcap\limits_{n=0}^{\infty} I_n$ contains at least one point and let it be α. but $\alpha \in [a, b]$

$\Rightarrow \quad \alpha \in \bigcap\limits_{n=0}^{\infty} I_n$

Since $f(x)$ is continous in $[a,b]$ and $\alpha \in [a,b] \Rightarrow f$ is continous at $x=\alpha$.

$\Rightarrow$ for every $\epsilon > 0$, $\exists\ \delta > 0$ st

$$|f(x) - f(\alpha)| < \epsilon \Leftrightarrow |x-\alpha| < \delta. \quad \text{---(i)}$$

let x_1 & $x_2 \in (\alpha-\delta, \alpha+\delta)$

then considr $|f(x_1) - f(x_2)|$

$$= |f(x_1) - f(\alpha) + f(\alpha) - f(x_2)|$$

$$\leq |f(x_1) - f(\alpha)| + |f(x_2) - f(\alpha)|$$

$$< \epsilon + \epsilon$$

let $\epsilon_1 = 2\epsilon$

$$\Rightarrow |f(x_1) - f(x_2)| < \epsilon_1 \Leftrightarrow |x-\alpha| < \delta.$$

we can choose n in sucha way that $[a_n, b_n]$ lies completely inside of $(\alpha-\delta, \alpha+\delta)$

$(\alpha-\delta \ldots [\ ]\ldots \alpha+\delta)$ a_n b_n

$\Rightarrow [a_n, b_n] \subset (\alpha-\delta, \alpha+\delta)$

$\Rightarrow |f(x_1) - f(x_2)| < \epsilon_1$ where $x_1, x_2 \in [\alpha-\delta, \alpha+\delta]$

$\Rightarrow$ This shows the theorm is true in $[a_n, b_n]$, which is contoary ~~But or~~ to our assumption.

$\Rightarrow$ our assumption was wrong

$\Rightarrow$ Theorem is true for same subintervals

H.P.